Propagation of Singularities
and Some Inverse Problems
in Wave Propagation

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Propagation of Singularities and Some Inverse Problems in Wave Propagation

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Abstract. We review a number of results relating the propagation of singularities for hyperbolic partial differential equations — i.e. the persistence, or non-localization, of wave motion — with well-posedness for some inverse problems of reflection type, such as arise for instance in seismology and ultrasonics. By far the most complete information is available for layered problems. We show how a simple but refined propagation-of-singularities theorem, with estimates, yields important functional properties of the model-data relationship for such problems, including regularity in various useful coefficient classes, separation of scales, ... We explain the essential role of *travel time* in the study of these problems, and show how its function may be generalized to multidimensional (i.e. non-layered) problems.

1. Introduction. The "inverse problems in wave propagation" of the title are idealizations of remote sensing techniques such as reflection seismology, ground-penetrating radar, and pulse-echo ultrasonic nondestructive evaluation. A wave of some sort is stimulated by an artificial energy source. It propagates into a region containing the remote structure of interest; the changes in physical properties through the structure cause some of the energy of the wave to be diverted into a reflected wave or echo, which travels in roughly the opposite direction, back to a region where the experimentalist has installed appropriate sensors and recording devices. The records of reflected waves are to be analysed to reveal whatever can be inferred about the physical properties of the remote structure.

Two conditions must be fulfilled for this scheme to work:

1. Energy, propagating in the form of waves, must penetrate the region of the remote structure;

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2. A reflection or echo process must take place, partitioning the energy of the wavefield.

The requirement that energy reach the target area in the form of waves (and reciprocally, return from the target area as waves) rather than as a diffusion process, say, is motivated by a need for spatial resolution in applications. Evidently the resolution scale for a wave process might be measured in wavelengths, whereas the resolution possible with measurements of diffusing fields is much harder to understand.

Roughly speaking, wave processes are modelled by hyperbolic systems of partial differential equations. The association is justified by the *geometrical optics analysis* of high-frequency solutions, which has its modern expression in various *propagation of singularities* results. These theorems give precise statements to the effect that a disturbance which begins coherently along a wavefront will continue in that form, the wavefront moving in such a way that its orthogonal trajectories (in the simplest cases) obey a system of ordinary differential equations. The description of the disturbance thus obtained corresponds well to the physical notion of *wave*, and appears to be correct when the coefficients of the hyperbolic system, which parameterize local physics of the propagating medium, are smooth on the scale of the wavelength of the disturbance.

In particular, when these coefficients are smooth on a scale much longer than a wavelength, the geometric optics approximation predicts *no reflected waves at all*. This prediction is borne out quite well by numerical experiments. Accordingly, models of reflecting media must involve coefficients rough on the scale of a wavelength, or finer. Morlet (1982) has made an extensive numerical study of the reflection and transmission of *plane waves in plane layered acoustic media* (these will be described below) — that is, the coefficients of the acoustic wave equation are functions of one variable only. Such plane-layered models are widely used in seismology, as for various reasons the mechanical properties of rocks often vary mostly in the vertical direction. Morlet found that once the length scale of strong variation in coefficients decreased to the level of a wavelength *and* the propagation distance remained constant, the wave nature of the propagating disturbance was strongly modified, and below some scale disappeared altogether: the wave did not propagate through such strongly oscillatory media at all. Since no "energy" penetrated the highly oscillatory region, there were no reflected waves out of which to infer its structure. That is, the wave was *localized* in the complement of the oscillatory region.

Thus models of media which are *too smooth* on the wavelength scale do not generate reflected waves because all of the "energy" in the wavefield passes through unaltered. Models which are *too rough*, on the other hand, generate no reflected waves because no "energy" penetrates to be reflected. ("Energy" here merely means some convenient measure of the local size of the field, which may or may not be related to a physical energy.) Thus the questions are naturally posed:

1. Do there exist models of a degree of roughness/smoothness *just right* to permit wave propagation *and* generate sufficiently informative reflected waves?

2. Do such models, if they exist, model real physical systems?

The first question is mathematical; partial answers will be sketched in the following pages. The second question does not have definitive answers, so far as the author is aware, though the motivation for question (1) is the *appearance* of experimental data from a variety of subjects. Clearer answers to question (2) await a combination of improvements in theory, numerical algorithms, and experimental technique.

While the generalities should be similar for the various physical models supporting wave propagation (electromagnetism, linear elastodynamics, ...), the details are best understood for linear acoustics, governed by the linearized pressure equation

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla u = F \text{ in } \mathbb{R}^n, \quad n = 1, 2, 3, \ldots$$

where $u(x,t)$ is the (scalar, infinitesimal excess) pressure field in a fluid, $\rho(x)$ is the density in the equilibrium state, $c(x)$ is the sound velocity, and the source term F is the divergence of a body force. We shall consider both

(i) F vanishes, and u has the form of a *plane wave*

$$u(x,t) = U(t - \theta \cdot x)$$

for large negative t — sensible when ρ and c are constant in a half-space, say (§§2–4);

(ii) u vanishes identically for large negative t, and F has point support, i.e.

$$F(x,t) = f(t)\delta(x - x_s) \qquad (\S 5)$$

The data of an inverse problem of the type considered here is the trace of the pressure field u on a time-like hypersurface — we shall consider only a coordinate hyperplane:

$$S[\rho, c; F] = u\big|_{x_n=0} \ .$$

Of course, in reality, measured signals are *sampled*, both spatially and temporally, and also *filtered* by the recording apparatus. In this paper we ignore such refinements.

Since the local "physics" of this model are determined entirely by ρ and c, the object of the inverse problem is to determine (to the extent possible) $\rho(x)$ and $c(x)$ for $x_n \neq 0$ from $S[\rho, c; F]$. (F will be regarded as known, though in fact it should be included in the unknowns for many models).

By far the best results are available for the plane-wave problem for layered media, i.e. $\rho = \rho(x_n)$, $c = c(x_n)$. In Section 2 we state a simple but quite precise propagation-of-singularities result for such problems, and exhibit the plethora of consequences which follow from it: regularity properties of the map S, rough correspondence between scales in u and scales in ρ, c, etc. We describe key results in the simplest form, and refer the reader to the literature for the most general versions. In §3 we show how these consequences of a refined propagation-of-singularity result can be used to investigate the plane-wave-inverse problem. The end result is a well-posedness result for "mildly rough" models: thus we are able to give an affirmative answer to question (1) above in this special instance. A very important

role is played in this development by the reparameterization of the layered model by travel-time: *without this device, essentially nothing can be accomplished, either theoretically or numerically.* In Section 4, we give a very brief treatment of rougher media, for which a dichotomy of scales exists, so that homogenization estimates can be used to reduce the problem to the mildly rough case.

The plane-wave/layered medium problems, while forming an excellent arena for development of insight, offer limited opportunity for immediate practical application. We consider the general multidimensional "point-source" problem (case (ii)) in §5. Very little is known about this class of problems, relative to the layered case. We address two issues:

(1) proper definition of the map S on non-smooth coefficient classes;

(2) location of a substitute for the travel-time transformation.

For both issues, we can offer only fragmentary information. We are able, however, to explain the function of the travel-time transformation for the plane-layered problems in such a way that its functional substitute for the non-layered problem is (essentially) uniquely specified. We give a construction of this "multidimensional travel-time transformation" under very restrictive smoothness assumptions, which (despite the restrictions) indicates that such things are actually possible — this is the only original result of the paper.

Most of the results and insights reported here are the result of joint work with many colleagues insofar as they are due to the author at all. Especially Fadil Santosa and Paul Sacks deserve much credit for whatever is worthwhile in the following pages. Special thanks are also due to Rakesh, R.M. Lewis, Cheryl Percell, and Gang Bao.

2. Some Simple Propagation-of-Singularities Results in Layered Media, and Consequences.

When the density ρ and sound velocity c are smooth and functions of only one space coordinate, say z, then the acoustic wave equation

$$\left(\frac{1}{\rho c^2} \frac{\partial^2}{\partial t^2} - \nabla \cdot \frac{1}{\rho} \nabla \right) u = 0 \tag{2.1}$$

has *plane-wave solutions*, i.e. the pressure $u(x, z, t)$ is in the form

$$u(x, z, t) = U(z, t - \xi \cdot x)$$

Such solutions exist for $\xi \in \mathbb{R}^n$ so long as $c(z)|\xi| < 1$ for all z. Evidently U is a solution of

$$\left[\frac{1}{\rho} \left(\frac{1}{c^2} - |\xi|^2 \right) \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial z} \frac{1}{\rho} \frac{\partial}{\partial z} \right] U = 0 \, . \tag{2.2}$$

Write $v = (c^{-2} - |\xi|^2)^{-\frac{1}{2}}$ and suppress ξ from the discussion for the remainder of this section. If $v \equiv v_0 \equiv$ const. for $z < 0$, and $a_0 \in \mathbb{R}$, then there exists a unique distribution solution to (2.2) with

$$U(z, t) = a_0 H(t - z/v_0)$$

for $t, z < 0$. Standard constructions, e.g. Courant and Hilbert (1962, Ch. V) imply that U has the progressing wave expansion

$$U(z,t) = a(z,t)H(t - \tau(z))$$

where

$$\tau(z) = \int_0^z \frac{1}{v}$$

is the *travel time*.

To pursue the agenda outlined in §1, one needs explicit estimates of a in terms of norms of ρ, c. Also, it will be advantageous to consider the slightly more general problem

$$\frac{1}{\rho v^2} \frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial U}{\partial z} \right) = \psi_0(z)\delta(t - \tau(z)) + \psi_1(z)w(z,t)H(t - \tau(z))$$

$$U = 0, \quad t << 0$$

with ψ_0, ψ_1 and w smooth. A simple calculation shows that we should take $U = aH(t - \tau)$ with a the solution of the characteristic Cauchy problem

$$\frac{1}{\rho v^2} \frac{\partial^2 a}{\partial t^2} - \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial a}{\partial z} \right) = \psi_1 w H(t - \tau) \quad \text{for } t > \tau(z) \, ;$$

$$\frac{d}{dz} a(z, \tau(z)) - \sigma(z)a(z) = \rho(z)v(z)\psi_0(z), \quad z > 0$$

$$a(0,0) = a_0 \tag{2.3}$$

where $\sigma(z) = \frac{d}{dz}(\log \rho v)(z))$. Following arguments in Symes (1986a; Section 1), we obtain estimates for the "sideways energy form"

$$Q_T(z) = \frac{1}{2} \int_{\tau(z)}^{2T - \tau(z)} dt \left(\left| \frac{\partial U}{\partial t} \right|^2 + \left| \frac{\partial U}{\partial z} \right|^2 \right)$$

of the form:

$$\|Q_T\|_{L^\infty[0, Z(Y)]} \leq$$

$$C \left(\|\psi_0\|_{L^2[0, Z(T)]}^2 + a_0^2 \|\sigma\|_{L^2[0, Z(T)]}^2 \right.$$

$$\left. + \|\psi_1\|_{L^2[0, Z(T)]}^2 \sup_{0 \leq z \leq Z(T)} \left\{ \int_{\tau(z)}^{2T - \tau(z)} dt |w(z,t)|^2 \right\} \right) \tag{2.4}$$

for suitable $C > 0$ depending *only* on $\|\rho\|_{H^1[0, Z(T)]}, \|v\|_{H^1[0, Z(T)]}$. Here

$$Z(T) = \inf \{ z : \tau(z) \geq T \} \, .$$

Since we also have control of the characteristic boundary condition $a(z, \tau(z))$ through (2.3), an identical bound follows for the H^1 norm of U in $L(T) = \{(z,t) : 0 \leq z \leq Z(T), \tau(z) \leq t \leq 2T - \tau(z)$. On the other hand, (2.4) is stronger than such a statement, and shows that the traces of first derivatives of U on the vertical line segments $\{z = \text{const.}\}$ are well defined.

Thus (2.4) is a rather strong propagation-of-regularity result — probably the strongest that can be expected to hold uniformly over ρ, v in $C^\infty \cap$ a bounded set in H^1_{loc}. It immediately allows us to extend the definition of various maps beyond the class of smooth coefficients. For example, suppose that $a_0 = 1$, ρ_1, ρ_2 both lie in the same bounded set in $H^1_{\mathrm{loc}} \cap C^\infty$, and $v \equiv 1$. The difference $V = U_2 - U_1$ satisfies

$$\frac{\partial^2 V}{\partial z^2} - \frac{\partial^2 V}{\partial z^2} + \sigma_1 \frac{\partial V}{\partial z} = (\sigma_2 - \sigma_1)\frac{\partial U_2}{\partial z}$$

where $\sigma_i = \frac{d}{dz} \log \rho_i$, $i = 1, 2$. From (2.4) with $\psi_0 = \psi_1 = w \equiv 0$ and $a_0 = 1$, we get a bound

$$
\begin{aligned}
\int_{\tau(z)}^{2T-\tau(z)} dt \left|\frac{\partial U_2}{\partial z}\right|^2 &\leq Q(z) \leq C\|\sigma_2\|^2_{L^2[0,Z(T)]} \\
&\leq C\|\rho_2\|_{H^1[0,Z(T)]}
\end{aligned}
$$

so we can use (2.4) again with $\psi_0 \equiv 0$, $\psi_1 = \sigma_2 - \sigma_1$, $w = \partial U_2/\partial z$, and $a_0 = 0$ to get

$$\int_{\tau(z)}^{2T-\tau(z)} dt \left(\left|\frac{\partial V}{\partial t}\right|^2 + \left|\frac{\partial V}{\partial x}\right|^2\right) \leq C\|\sigma_2 - \sigma_1\|^2_{L^2[0,Z(T)]}$$

whence a Lipschitz estimate follows for the map

$$\tilde{S}_\delta : \log \rho \in H^1[0, Z(T)] \mapsto \frac{\partial U}{\partial t}(0, \cdot) \in L^2[0, 2T]$$

for example, uniform over bounded sets in $H^1[0, Z(T)] \cap C^\infty(\mathbb{R})$. Consequently the map extends — in some sense, we have "solved" this singular problem with non-smooth (H^1) coefficients.

In Symes (1986a) these arguments are developed further to show that $\tilde{S}_\delta$ is actually a C^1 diffeomorphism and further work along the same lines shows that $\tilde{S}_\delta$ is C^2. It is probably C^∞. Since $\tilde{S}_\delta$ is differentiable, estimates for the derivative are useful. It is possible to show, by very similar arguments, that the formal linearization

$$\left(\frac{1}{\rho v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial z}\frac{1}{\rho}\frac{\partial}{\partial z}\right)\delta U = -\frac{\partial \sigma}{\partial z}\frac{1}{\rho}\frac{\partial U}{\partial z} \qquad \delta U \equiv 0, \ t << 0$$

actually defines the derivative

$$D\tilde{S}_\delta[\rho]\delta\rho = \frac{\partial \delta U}{\partial t}(0, \cdot).$$

Moreover,

$$C_*\|\delta\rho\|_{H^1[0,Z(T)]} \leq \|D\tilde{S}_\delta[\rho]\delta\rho\|_{L^2[0,Z(T)]} \leq C^*\|\delta\rho\|_{H^1[0,Z(T)]}$$

where the constants $0 < C_* \leq C^*$ are uniform as $\log \rho$ ranges over bounded sets in H^1_{loc}. Similarly,

$$D\tilde{S}_H[\rho]\delta\rho = \delta U(0, \cdot)$$

satisfies

$$\|D\tilde{S}_H[\rho]\delta\rho\|_{L^2[0,2T]} \leq C_H\|\delta\rho\|_{L^2[0,Z(T)]}$$

with another H^1_{loc}-uniform constant C_H, though no lower estimate seems to obtain for $D\tilde{S}_H$.

These estimates show that there is *approximate separation of scales* in the relation between coefficients and solution, in the following sense. A convolution kernel $f \in \mathcal{E}'$ is *elliptic* on H^1_{loc} if there exist constants K_0, K_1, K^* so that for $u \in H^1(\mathbb{R})$,

$$K_1\|u\|_1 - K_0\|u\|_0 \leq \|f * u\|_1 \leq K^*\|u\|_1 .$$

For example, if $|\hat{f}(\omega)| \geq p$ for $|\omega| \geq \Omega_*$, then one could take crudely $K_1 = p, K_0 = (1 + \Omega_*^2)^{\frac{1}{2}} p$ (much more precise statements are possible). Thus the relation between the constants K_0, K_1 measures the threshold of the "passband" of f. Such f may be manufactured by subtracting from the Dirac δ function a smooth approximation. Without loss of generality we may assume that f is "causal", i.e. supp $f \subset [0, \infty)$.

As a consequence of the above estimates for $D\tilde{S}_\delta$, the derivative of

$$\tilde{S}_f = f * \tilde{S}_\delta$$

satisfies

$$\begin{aligned}
\|D\tilde{S}_f[\rho]\delta\rho\| \geq \; & \tilde{K}_1\|\delta\rho\|_{H^1[0,Z(T)]} - \tilde{K}_0\|\delta\rho\|_{L^2[0,Z(T)+Z_f]} \\
& - \tilde{K}_m m_1(f)\|\delta\rho\|_{H^1[Z(T),Z(T)+Z_f]}
\end{aligned} \tag{2.5}$$

where $\tilde{K}_1, \tilde{K}_0$ and $\tilde{K}_m$ depend on $\|\log\rho\|_{H^1[0,Z(T)]}$, the constants K^*, K_1 and K_0 in the ellipticity estimate of f, and K_m and Z_f depend also on the *first moment*

$$m_1(f) = \inf_{t_0 \in \mathbb{R}} \sup_{\phi \in C_0^\infty(\mathbb{R})} \frac{\langle f, (t - t_0)\phi \rangle}{\|\phi\|_{L^\infty}}$$

of f.

The first two terms in the preceding estimate show that $D\tilde{S}_f$ has essentially the same "filter" or separation-of-scales effect as does $f*$, at least qualitatively. The last term is necessary because the "breadth" of f, as measured by its first moment $m_1(f)$, may be nonzero, so that control of $\delta\tilde{S}_f$ is needed over a slightly longer time interval — or equivalently of $\delta\rho$ over an "extra" depth interval — to ensure control over $\delta\rho$ on $[0, Z(T)]$. Both the coefficient $K_m m_1(f)$ and the size Z_f of the extra depth interval are $O(m_1(f))$.

In order to deal with truly "bandlimited" kernels — i.e. $f \in C_0^\infty$ as well — higher-order estimates are needed. Suzuki (1988) showed (essentially) that $\tilde{S}_{\delta'} = \partial/\partial t \tilde{S}_\delta$ obeys an estimate

$$\|D\tilde{S}_{\delta'}[\rho]\delta\rho\|_{L^2[0,2T]} \leq C_2\|\delta\rho\|_{H^2[0,Z(T)]}$$

using extensions of the arguments sketched above; the constant is uniform over bounded sets in H^2_{loc}. This estimate can be used to study $\tilde{S}_f$ when $f*$ obeys an estimate of the form

$$K_1\|u\|_1 - K_0\|u\|_0 - K_2\|u\|_2 \leq \|f * u\|_1 \leq K^*\|u\|_1 . \tag{2.6}$$

Once again, this estimate can be interpreted loosely as identifying the "passband" of f. If $|\hat{f}(\omega)| \geq p > 0$ for $\Omega_* \leq |\omega| \leq \Omega^*$, one can take $K_1 = p$, $K_0 = p(1 + \Omega_*)^{\frac{1}{2}}$ as before and $K_2 = p(1 + \Omega^*)^{-\frac{1}{2}}$. There results for $D\tilde{S}_f$ an estimate of the form

$$
\begin{aligned}
\|D\tilde{S}_f[\rho]\delta\rho\|_{L^2[0,2T]} \geq\ & \tilde{K}_1\|\delta\rho\|_{H^1[0,Z(T)]} - \tilde{K}_0\|\delta\rho\|_{L^2[0,Z(T)]} \\
& - \tilde{K}_2\|\delta\rho\|_{H^2[0,Z(T)]} \\
& - \tilde{K}_m m_1(f)\|\delta\rho\|_{H^1[Z(T),Z(T)+Z_f]}
\end{aligned}
$$

all dependences being as before, except for $\tilde{K}_2$ which is

(a) uniform over bounded sets in H^2_{loc} for fixed f;

(b) for ρ in a fixed bounded set in H^2_{loc}, $\tilde{K}_2 = O(K_2)$.

Thus the estimate is nontrivial if $\tilde{K}_2$ is small enough, i.e. if f is "close enough to elliptic" relative to $\|\rho\|_{H^1[0,Z(T)]}$.

To obtain similar results for the map $v \mapsto U$, or $(\rho, v) \mapsto U$, one must impose much more smoothness on v, essentially because a change in v actually moves the characteristics. See e.g. Lewis's thesis (1989) or Symes (1986b). The diffeomorphism is lost: in metrics strong enough to make $v \mapsto U$ differentiable, the derivative is not coercive.

3. Travel Time and the Determination of "Mildly Rough" Coefficients. Maintain the setting of the previous section, but now consider, for simplicity, the case $\rho \equiv \text{const.} (= 1)$ instead. Introduce the (*travel-time*) change-of-variable

$$
x = \tau(z)
$$

(NOTE: we are using "x" in a different sense than at the beginning of §2!) Denoting by ζ the inverse function of τ, so that

$$
x = \int_0^{\zeta(x)} \frac{1}{v}
$$

introduce

$$
\tilde{v}(x) = v(\zeta(x)), \quad \tilde{U}(x,t) = U(\zeta(x),t) .
$$

A short calculation verifies that

$$
\tilde{U}(x,t) = H(t-x), \qquad x,t < 0 ;
$$

$$
\frac{\partial^2 \tilde{U}}{\partial t^2} - \frac{\partial^2 \tilde{U}}{\partial x^2} - \tilde{\sigma}\frac{\partial \tilde{U}}{\partial x} = 0
$$

where $\tilde{\sigma} = \frac{d}{dx}\log\tilde{v}$. Thus $\tilde{v}$ plays exactly the same role as ρ did at the end of the last section, hence the map

$$
\tilde{S}_\delta : \tilde{v} \in H^1[0,T] \to \frac{\partial \tilde{U}}{\partial t}(0,\cdot) \in L^2[0,2T]
$$

is well-defined by continuous extension from $\tilde{v} \in C^\infty$, is a C^2 diffeomorphism, and all the "quasi-elliptic" estimates in §2 apply ipso-facto to $\tilde{S}_f = f * \tilde{S}_\delta$.

Remark In §5 some commentary is offered on the "real" significance of this change of variable.

To understand how results about $\tilde{S}_f$ might yield information about v (rather than $\tilde{v}$), examine the change-of variables operator:

$$M[v] = v \circ \zeta .$$

It is possible to show that

    (i) M extends to a continuous map defined on the positive cone

$$\Gamma_k = \{v \in H^k_{\text{loc}}[0, \infty) : v(0) = v_0, v(z) > 0 \; \forall z\}$$

    and taking values in Γ_k, for $k \geq 1$;

    (ii) M is nowhere locally uniformly continuous, hence *a fortiori* not differentiable, in Γ_k.

    (iii) Regarded as a map: $\Gamma_{k+2} \to \Gamma_k$, M is differentiable.

For discussion and proofs of related results see Lewis (1989) and Symes (1986b). The continuity result is due to Rakesh (unpublished).

Formally, the derivative of M is given by

$$\begin{aligned} \delta M(t) &= \delta v(\zeta(t)) + v'(\zeta(t))\delta\zeta(t) \\ &= \left. \left(\delta v + vv' \int_0^z \frac{\delta v}{v^2} \right) \right|_{z=\zeta(t)} . \end{aligned}$$

Thus the "principal part" of δM is the change-of-variables for v, applied to δv — the simplest sort of Fourier integral operator. In fact, if we assume that $v = $ const. for large Z, then

$$\frac{d}{dz} M[v] = \int dk \, a(t,k) e^{i\zeta(t)k} \left(\frac{dv}{dz} \right)^\wedge (k)$$

where the phase ζ itself depends on v — i.e. dM/dz is a "nonlinear Fourier integral operator."

On the other hand, the appearance of v' in the second term above hints at the nondifferentiability of $M : \Gamma_k \to \Gamma_k$.

Now suppose that f is an elliptic convolution kernel, as in the previous section. Appraisal of the "Gårding" estimate (2.5) shows that, if $\delta\tilde{v}$ is sufficiently smooth — that is, $\|\delta\tilde{v}\|_{L^2}$ is sufficiently big relative to $\|\delta\tilde{v}\|_{H^1}$ — the estimate has no force. If $\delta\tilde{v} = \delta M$, with smooth δv, however, it may be that δv (rather than $\delta\tilde{v}$) is controlled. In fact, if $\|v'\|_{H^1}$ is large, one might suspect that the second term in the expression for δM dominates $\| \int \frac{\delta v}{v^2} \|_{L^2}$, hence (for smooth δv) $\|\delta v\|_{L^2}$ itself. This idea is embodied in the "rough/smooth" lemma (see Symes (1988), Lemma 3):

For $u, \Phi \in C^\infty$, $a < b$, $\Delta > 0$, set

$$r(x, \Delta) = \frac{1}{\Delta} \int_{x - \frac{\Delta}{2}}^{x + \frac{\Delta}{2}} |u|^2$$

$$r_*(\Delta) = \inf_{x \in [a,b]} r(x, \Delta), \quad r^* = \sup_{x \in [a,b]} r(x, \Delta)$$

Then

$$\|\Phi u\|_{L^2[a,b]}^2 \geq \frac{r_*(\Delta)}{2} \|\Phi\|_{L^2[a,b]}^2$$

$$- \frac{16}{9}(r_*(\Delta) + r^*(\Delta))\Delta^2 \|\Phi'\|_{L^2[a,b]}^2 \qquad (3.7)$$

Apply this estimate to the second term in δM: with $u = \left(\frac{1}{2}v^2\right)''$, $\Phi = \int \cdot \frac{\delta v}{v^2}$, we get

$$\left\|\left(\delta v + vv' \int \frac{\delta v}{v^2}\right) \circ \zeta\right\|_{H^1[0,T]}^2 \geq C_1 r_*(\Delta) \left\|\int \frac{\delta v}{v^2}\right\|_{L^2[0,Z(T)]}^2 - C_2 \|\delta v\|_{H^1[0,Z(T)]}^2$$

$$- C_3(r^*(\Delta)\Delta^2 + 1) \|\delta v\|_{L^2[0,Z(T)]}^2$$

where C_1, C_2, and C_3 are uniform for v in bounded sets in H^1_{loc}. Thus if δv is smooth — so that $\|\delta v\|_{H^1}$, $\|\delta v\|_{L^2}$ and $\|\int \delta v\|_{L^2}$ are comparable — *and if v'' is uniformly big relative to $\|v\|_{H^1}$, locally on a length scale of Δ, and Δ is small enough, — then*

$$\|\delta M\|_{H^1[0,T]} \geq C \|\delta v\|_{L^2[0,Z(T)]} . \qquad (3.8)$$

Define $S_f = \tilde{S}_f \circ M$. Combining (3.8) with (2.5) we see that if in addition $\|\delta v\|_{L^2}$ is big enough relative to $\|\delta v\|_{H^1}$, then

$$\|\delta S\|_{L^2[0,2T]} \geq C \|\delta v\|_{L^2[0,Z(T)]}$$

(presuming $\delta v \equiv 0$ in $(Z[T], Z[T] + Z_f)$ in order not to have to worry about the "margin correction"). That is, δS is coercive, in a certain sense, for very smooth δv. On the other hand it is plain that δS is coercive for oscillatory δv.

Unfortunately, that's as far as the purely 1-D problem goes. If δv is neither purely smooth nor purely oscillatory, it is possible to arrange for the two parts of δM to very nearly cancel, even when v satisfies the "uniform roughness" condition sketched above, so that δS cannot possibly be coercive.

The way out is to remember — from the beginning of §2 — that each choice of sound speed profile $c(z)$ (keeping ρ constant still) gives a *suite* of one-dimensional problems with plane-wave velocities

$$v(z, \xi) = c(z)(1 - c^2(z)\xi^2)^{-\frac{1}{2}} .$$

The change-of-variables operator M and the seismogram operators S_f thereby acquire dependence on ξ as well, of course. The perturbation in v is related to the perturbation in c by

$$\delta v = \delta c(1 - c^2 \xi^2)^{-\frac{3}{2}}$$

so

$$\delta M = (1 - c^2\xi^2)^{-\frac{3}{2}} \left[\delta c - cc'(1 - c^2\xi^2)^{-\frac{1}{2}} \int^z \frac{\delta c}{c}(1 - c^2\xi^2)^{-\frac{1}{2}} \right] \Bigg|_{z=\xi}.$$

It is not surprising that, while δM can be made to vanish at *one* value of ξ, it *cannot* be made to vanish over a range of ξ simultaneously. In fact, a coercivity estimate is possible of the form

$$\|\delta M\|_{H^1([0,T]\times[\xi_{min},\xi_{max}])} \geq C\|\delta c\|_{L^2[0,Z(T)]}$$

when C is "rough enough."

The preceding is only the barest sketch of an argument, of course. In Symes (1988) we have shown that all of the choices mentioned above can actually be made. The end result is the identification of a class $\Sigma \subset H^1_{loc}$ of velocity profiles c and of suitable "elliptic" kernels f which permit stable solution of the inverse problem. More precisely, for each elliptic kernel f, there exists a neighborhood V_f of $S_f(\Sigma)$ in L^2_{loc} and a locally Lipschitz map $I_f : V_f \to L^2_{loc}$ satisfying

$$I_f(S_f(c)) = c \quad \text{for} \quad c \in \Sigma .$$

When f is "bandlimited," i.e. smooth but obeying a quasi-elliptic estimate like (2.6), a similar result is obtained *via* regularization: if the "upper bandlimit" parameter K_2 is small enough, a stable inverse map I_f exists for which $[I_f(S_f(c)) - c]$ is small — as usual with regularization, one does not obtain exact recovery of the model. The degree of error (for consistent data in $S_f(\Sigma)$) can be made as small as one likes, at the usual price of degradation of stability.

The construction of I depends on a variational problem of least-squares type. This variational problem has served as the basis of a computer program to solve the inverse problem numerically, which has yielded quite promising results when applied to both synthetic and field seismic data. For details see Symes (1988), Carazzone and Symes (1989), and references cited there.

4. Very Rough Models and Homogenization. The sedimentary crust, amongst other real materials, does not appear to possess the smoothness relative to wavelengths and other scales necessary to make plausible direct application of the results of the last section. Instead, very rough models may be *approximated* in a weak sense by smooth ones, provided that a dichotomy of length scales is present.

For 1-dimensional wave propagation, a version of such an approximation scheme may be based on the quantitative homogenization estimates developed by Bamberger *et al.* (1979). Specialized to the present context, this estimate amounts to

$$\|S_f[c_1] - S_f[c_2]\|_{L^2[0,2T]} \leq C\|f\|_{H^4(\mathbb{R})}\Gamma(c_1, c_2)$$

for $\log c_1, \log c_2 \in L^\infty_{loc(\mathbb{R})}$. The pseudo-distance Γ is defined by

$$\Gamma(c_1, c_2) = \sup_{0 \leq z \leq Z} \left| \int_0^z dz' \left(\frac{1}{c_1^2(z')} - \frac{1}{c_2^2(z')} \right) \right|$$

where Z is chosen sufficiently large.

If Σ is the set of models described in the previous section, V_f the neighborhood of $S_f(\Sigma)$, and f is a "quasi-elliptic" but smooth kernel, we can associate to the pair (Σ, f) the set Σ_f of $\log c \in L^\infty_{loc}$ for which $S_f[c] \in V_f$. It is evident from the definition that Σ_f contains quite a bit more than an L^2-neighborhood of Σ. An easy way to see this is to pick a Dirac kernel $\phi \in C^\infty_0(\mathbb{R})$, $\phi(0) \neq 0$, $\int \phi = 1$, and set for $\epsilon > 0$ $\phi_\epsilon(z) = \epsilon^{-1}\phi(\epsilon^{-1}z)$. Roughly, ϵ is the "width" of ϕ_ϵ, and the convolution with ϕ_ϵ *averages* on the length scale ϵ. Then for $\log c \in L^\infty_{loc}$

$$\Gamma(c, \phi_\epsilon * c^{-2})^{-\frac{1}{2}} = O(\epsilon) \ .$$

Thus $c \in \Sigma_f$ if $\phi_\epsilon * c \in \Sigma$ for ϵ small enough that $S_f[c] \in V_f$. Such c can have arbitrarily large variation, provided that it is on a sufficiently fine length scale that averaging over length scale $\epsilon = O(\|f\|_{H^4}^{-1})$ yields an element of Σ. Σ is characterized also by being a subset of a large ball in H^2, the radius being determined by the "upper bandlimit" of the kernel (i.e. K_2 in 2.6). Thus the scales permitted in Σ are determined by this upper bandlimit. All significant components at shorter scales must be removed (essentially) by averaging on the length scale ϵ. Thus the difference $c - (\phi_\epsilon * c^{-2})^{-\frac{1}{2}}$ must be oscillatory on a scale smaller than ϵ, and much smaller than that permitted by membership in Σ.

These choices of scale can be made precise. In an unpublished technical report, the author made most of these choices explicit and investigated the question numerically. The results lent credence to the heuristic rule:

> *For effective recovery of the layered medium, the velocity profile must consist of a "core" medium which oscillates on a wavelength scale, and perhaps a very rough component oscillating on a subwavelength scale.*

Such a clean frequency dichotomy is probably not seen in nature; in fact some evidence exists that actual velocity distributions are *fractal*. Thus oscillatory components exist at all scales, and the preceding heuristic condition is not satisfied. It is not clear whether the "mildly rough" results bring with them results about media without the frequency dichotomy described here.

Papanicolau, White, and colleagues have studied another class of media admitting a dichotomy of length scales (Burridge *et al.* (1989)). The principal difference in problem setting between our work and theirs is in the relation between the source frequency content and the length scale of oscillation in the coefficients. Our work pertains to a fixed source, hence fixed bandwidth, and the limit as the length scale for the coefficients goes to zero. White *et al.* consider instead a family of sources with frequency increasing roughly as the square-root of the frequency of oscillation in the parameters (after normalization to fixed propagation time). Thus in their work the oscillations in the coefficient continue to interact with the oscillatory energy in the solution, which does not approach that of a smooth-coefficient equation. It is not clear which asymptotic regime is more important in explaining practical phenomena. Most likely both regimes are present in real data.

5. Multidimensional Problems.

We return to the problem specified at the beginning of Section 2, viz. linear acoustics, but this time in dimension $n > 1$ and

with the coefficients allowed to depend on all the space variables. We introduce "energy" into the system *via* a "point source," i.e. a right-hand side with point support in x:

$$\left(\frac{1}{\rho(x)c^2(x)}\frac{\partial^2}{\partial t^2} - \nabla \cdot \frac{1}{\rho(x)}\nabla\right)u(x,t) = f(t)\delta(x) \qquad u \equiv 0, \ t < 0 \ .$$

Once again we are interested in estimates which hold for elliptic f, uniformly over bounded sets of coefficients in Sobolev spaces and especially in estimates for time-like traces.

Very little is known about these problems, and the following discussion is meant mainly to illustrate some of the remaining difficulties. Some partial results may be found in Rakesh (1988), Sacks and Symes (1985), and Sun (1987). Two major differences emerge compared to the layered case. The first of these is that considerably more coefficient regularity is needed to assure the optimal regularity of timelike traces. To fix ideas, take $f = \delta^{(-\frac{n-1}{2})}$ above, and assume that $\log c \in C^\infty$ is fixed. Near $x = t = 0$, Hadamard's construction (Courant and Hilbert, Ch. 6) yields

$$u(x,t) = a(x)\rho^{-\frac{1}{2}}(x)H(t - \tau(x)) + R_0(x,t)$$

where $\tau(x)$ is the geodesic distance from x to 0 in the metric $c^{-2}\sum_i dx_i dx_i$, a is a C^∞ transport coefficient, and R_o vanishes at $t = \tau(x)$. This expression remains correct so long as we encounter no point conjugate to 0, i.e. as long as no caustics appear in the wave-front, which we assume (the complication in this analysis arising from caustics have not been understood, nor has the effect of changing c — see some discussion below). Thus u may as well be viewed as the solution of an interior characteristic Cauchy problem.

It is natural to attempt to bound the *energy norm*: With

$$\begin{aligned}
B_T &= \{x \ : \ \tau(x) \le T\} \\
C_T &= \{(x,t) \ : \ t = \tau(x) \le T\}
\end{aligned}$$

we get the standard energy identity

$$\int_{B_T} dx \left[\frac{1}{\rho c^2}\left|\frac{\partial u}{\partial t}\right|^2 + \frac{1}{\rho}|\nabla u|^2\right] = \int_{C_T}\frac{1}{\rho}\left|\frac{\partial u}{\partial t}\nabla\tau + \nabla u\right|^2 \ .$$

The transport equation (Hadamard's construction, above) yields all of the derivatives of u *tangential* to C_T, but the normal derivative is also needed to evaluate the R.H.S. of the energy identity. The normal derivative — or equally well, the t-derivative — trace on C_T is determined by the second transport equation,

$$\nabla\tau \cdot \nabla a_1 + (b + \nabla\log\rho \cdot \nabla\tau)a_1 =$$
$$(\Delta - \nabla\log\rho \cdot \nabla)(a\rho^{-\frac{1}{2}})$$
$$\lim_{t \to \tau(x)+}\frac{\partial u}{\partial t}(x,t) = a_1(x) \ .$$

Here b depends on the bicharacteristic geometry, hence on c. Thus second derivatives of ρ are involved, and will not be "integrated out," an important difference with the layered case.

A certain amount of evidence suggests that the proper way to resolve this and other problems is to impose *microlocal* restrictions on the coefficients. Fundamentally, all of this evidence rests on Rauch's Lemma (Rauch, 1979), which we state here in a lopsided form suitable for application to linear problems, using now-conventional notation (Beals and Reed, 1984):

> *Suppose that for some* $(x_0, \xi_0) \in T^*\mathbb{R}^n$, *the distributions* a, u *satisfy* $a \in H^s \cap H^r_{m\ell}(x_0, \xi_0)$, $u \in H^t \cap H^q_{m\ell}(x_0, \xi_0)$ *with* $0 \le t \le s$, $q \le t + r - s$, $q < s + t - n/2$. *Then* $a\,u \in H^t \cap H^q_{m\ell}(x_0, \xi_0)$.

That is, $H^t \cap H^q_{m\ell}(x_0, \xi_0)$ is an $(H^s \cap H^r_{m\ell}(x_0, \xi_0))$–algebra. The lemma is usually stated for $t = s$, $q = r$, as is appropriate for applications to nonlinear problems. For linear problems such as that considered here, the solution may well have a different degree of smoothness than the coefficients — in fact, the solution u above is expected to have a jump discontinuity. Such strong singularities are much more difficult to treat in nonlinear problems, and most of the results based on Rauch's lemma and related ideas concern much weaker singularities — see e.g. Beals and Reed (1982, 1984). Nonetheless, we expect to be able to understand the relation between the coefficient and solution regularity *away* from the leading singularity or wavefront. For example, a quantified version of Rauch's lemma enables one to prove microlocal energy estimates for first-order hyperbolics like the transport equation above. In the present instance, we obtain $a_1 \in H^s$ if $\rho \in H^{s+2}$, $s > 1 + n/2$ (n = number of *space* dimensions) plus microlocal conditions. Then Theorem 2.1 in Beals and Reed (1984) implies that $a_1 \in H^{s+1}_{m\ell}$ away from the characteristic variety of the transport operator. One then iterates this construction to produce higher energy estimates for u interior to the light cone.

Another piece of evidence in favor of microlocal coefficient classes is the following trace theorem due to Bao (1989): it concerns the case $c \equiv 1$.

> *Suppose* $s > 3 + n/2$ *and*
>
> *(i)* $\left[\frac{\partial^2}{\partial t^2} - \Delta - \nabla\sigma \cdot \nabla\right] u = f$ *in* $\mathbb{R}^{n+1}$
>
> *(ii)* $u, f \equiv 0$ *for* $t << 0$
>
> *(iii)* Γ *is a closed conic set containing*
> $\{(x, \xi) : |\xi_n| \le \epsilon|\xi'|, \ |x| \le \epsilon\}$
> *for small* $\epsilon > 0$ *(here* $\xi' = (\xi_1, \ldots \xi_{n-1})$*)*
>
> *(iv)* $u, \nabla\sigma$, *and* $f \in H^{s-1} \cap H^s_{m\ell}(\Gamma)$
>
> *Then for* $\phi \in C_0^\infty(\mathbb{R}^{n+1})$,
> $$\phi u\big|_{x_n=0} \in H^s .$$

Thus traces are as regular as the solutions, and particularly as regular as the coefficients allow — but now at a higher degree of smoothness than in the layered case, and only with microlocal restrictions against tangential oscillations.

A great deal remains to be said along these lines, but at least one can imagine that the problems are in some sense "technical." The very delicate dependence of the solution on the principal part of the wave operator — physically on the velocities — poses a quite different sort of difficulty, which forms the second topic of this section. For layered media and plane-wave solutions, the travel-time transformation was available to regularize this dependence — by making the coefficients of the principal part constant! No such possibility exists for $n > 1$. To see how the essential role of the travel-time transformation may nonetheless be filled, we examine once more the one-dimensional case.

Recall from Section 3 that we could write

$$M[r](t) = \int dk \, a(t,k) e^{i\zeta[r](t)k} \hat{r}(k)$$

where $r = dv/dz$ and is presumed to have compact support, and $\zeta[r]$ is the corresponding inverse travel-time function.

Consider more generally a family of oscillatory integral operators of the form

$$G[u]\phi(x) = \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^N} d\theta \, a(u; x, y, \theta) e^{i\phi(u; x, y, \theta)} \phi(y)$$

where the symbol a and phase ϕ are Gateaux-differentiable functions of $u \in C_0^\infty(\mathbb{R})$, to begin with, and assume that G is elliptic. From the calculus of such operators, it follows that the formal perturbation

$$
\begin{aligned}
\delta G[u; \delta u]\phi(x) &= \int dy \int d\theta \, (\delta a(u, \delta u; x, \theta) \\
&\quad + ia(u; x, \theta)\delta\phi(u, \delta u; x, y, \theta)) \, e^{i\phi(u; x, y, \theta)} \phi(y)
\end{aligned}
$$

is of order higher than that of G, and can be expressed as

$$\delta G[u, \delta u] = G[u]P[u, \delta u] = \tilde{P}[u, \delta u]G[u]$$

where $P, \tilde{P}$ are elliptic pseudo differential operators of order > 0 with symbols linearly dependent on δu. The travel-time transform actually has the form

$$u \mapsto G[u]u \ .$$

Such operators can be said to "have" a canonical relation, even though they are nonlinear — namely, that of $G[u]$. When u changes the canonical relation changes, shifting (infinitesimally: differentiating) the singularities of the image. In order that δG may have the same sort of bounds as G — in terms of δu and of u — it is necessary to compose G with another (necessarily equally irregular) transform, having the inverse canonical relation. That is, we need to find H and Q so that $Q = G \circ H$ has a derivative δQ with the same qualitative estimates as Q itself. It is natural to look for H also in the form of an oscillatory integral, and to expect Q to be pseudo-differential — this is exactly what happens in the case of travel-time. That is, we seek $H[\tilde{u}]$ so that

$$Q[\tilde{u}] = G[H[\tilde{u}]\tilde{u}]H[\tilde{u}]$$

is pseudodifferential, with symbol depending smoothly on $\tilde{u}$.

In the case at hand, $H[\tilde{u}]$ is essentially uniquely specified by this requirement, since its canonical relation is specified. To construct H explicitly, it is necessary to explain first the canonical relation associated to the seismogram operator. Fortunately, the *linearized* seismogram operator has the same canonical relation, and this latter has been discussed extensively — see e.g. Beylkin (1985), Rakesh (1988), for example. The matter is clearest when the oscillatory integral G is expressed as a generalized Radon transform (the viewpoint emphasized by Beylkin).

For simplicity we discuss only the two-dimensional case. Let $R = \{(x_1, x_2) : -X \le x_1 \le X, 0 \le x_2 \le Z\}$ and suppose that C is a class of C^1 metrics of the form $c^{-2} \sum_i dx_i dx_i$ in R for which the geodesics between points in R and points in the surface segment $[-X_R, X_R] \times \{0\}$, $x_R < X$, have no conjugate points and lie entirely in R. This hypothesis holds for $c \equiv$ const., hence for a C^2–neighborhood of the constants. This "no-caustics" hypothesis is necessary if results as simple as those to follow are to hold, as is shown in Percell's thesis (1989).

Denoting by $\tau(x, y)$ the geodesic distance from x to y, the *reflection phase* for $y \in R$, $x_r \epsilon [-X_R, X_R]$:

$$\phi(c; x_r; y) = \tau(0, y) + \tau(y, (x_r, 0))$$

is well-defined for $c \in C$. For $f \in C^\infty(R)$ define

$$G[c]f(x_r, t) = \int_{y:\phi(c;x_r,y)=t} f \ .$$

That is, $G[c]f$ is the generalized Radon transform of f over the equal-time curve $\{y : \phi(c; x_r, y) = t\}$ generated by C. For $c =$ const., the equal-time curves are arcs of ellipses with focii at $(0, 0), (x_r, 0)$ and radii proportional to t.

For smooth c, the impulsive seismogram operator

$$S[c] = u|_{x_2=0}$$

where

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \Delta u = \delta(t)\delta(x) \qquad u \equiv 0, \ t < 0$$

can be expressed as

$$S[c] = \tilde{p}[c]G[c]c$$

for a suitable pseudodifferential $\tilde{p}[c]$, so S "has the same canonical relation as G."

The class of operators with the inverse canonical relation to that of $G[c]$ all differ from the weighted adjoint operator

$$G^*[c]\tilde{v}(y) = \int_{-X_R}^{X_R} dx_r b(x_r, X_R, y)\tilde{v}(x_r, \phi(c; x_r, y))$$

by composition with the pseudodifferential operator.

Note that $G^*[c]$ maps functions $\tilde{v}(x_r, t)$ of "receiver" and "time" to functions of y, i.e. space — in exploration seismic parlance, $G^*[c]$ is a *migration operator*. The weight b has been included for generality — changing it will change the result by a pseudodifferential operator.

Since $G^*[c]$ depends on c, it is not exactly right. We need an operator defined in terms of functions $\tilde{c}(x_r, t)$, which will function as the equivalent of the travel-time velocity $\tilde{v}$ of Section 3. We define such an operator in terms of an auxiliary problem, as follows.

The "trick" is to construct directly the reflection phase. Note that

$$\phi(1; x_r, y) = |y| + |y_1 - x_r, y_2)| =: \Phi_0(x_r, y) .$$

We will need to control first derivatives of the reflection phase — but Φ_0 is not C^1 near $y = 0$, $y = (x_r, 0)$. So we define instead for $\delta > 0$, $k = 1, 2, \ldots$

$$
\begin{aligned}
\mathcal{P}_\delta^k \;=\; \Big\{ &\Phi \in C^0([-X_R, X_R] \times R) : \\
&\Phi(x_r, y) = \Phi_0(x_r, y) + \Delta\Phi_1(x_r, y), \\
&\Delta\Phi_1 \in C^k([-X_R, X_R] \times R) \ \text{ and} \\
&\Phi_1(x_r, y) = 0 \ \text{ if } y_2 < \delta \,.\Big\}
\end{aligned}
$$

It is easy to see that if $c \in \mathcal{C}$ satisfies $c(y) \equiv 1$ for $y_2 < \delta$, then $\phi(c; \cdot, \cdot) \in \mathcal{P}_\delta^1$. We give $\mathcal{P}_\delta^k$ the C^k topology in the obvious way.

Pick $\gamma \in C^\infty(\mathbb{R})$ with $\gamma(y) \equiv 0$ for $y_2 < \delta$, $\gamma(y) \equiv 1$ for $y_2 > 2\delta$. For $\tilde{r} \in C^3([-X_R, X_R] \times \mathbb{R}^+)$, $\Phi \in \mathcal{P}_\delta^k$ define

$$
\begin{aligned}
\Phi_*(\tilde{r}) \;&=\; \int_{-X_R}^{X_R} dx_r \, \tilde{r}(x_r, \Phi(x_r y)) \\
v[\Phi, \tilde{r}] \;&=\; 1 + \gamma \Phi_*[\tilde{r}]
\end{aligned}
$$

We consider the fixed-point problem

$$T_{\tilde{r}}\Phi := \phi(v[\Phi, \tilde{r}]; \cdot, \cdot) = \Phi .$$

If $\Phi = \Phi[\tilde{r}]$ solves this fixed-point problem, then the operator Φ_* has the same canonical relation as $G^*[v[\Phi, \tilde{r}]]$, hence the inverse canonical relation to $G[v[\Phi, \tilde{r}]]$ or $S[v[\Phi, \tilde{r}]]$. Hence

$$\tilde{r} \mapsto S[v[\Phi[\tilde{r}], \tilde{r}]]$$

is "nonlinear pseudodifferential" and has some hope of extending to a differentiable map with coercive derivative on interesting classes.

Note that if $\|\tilde{r}\|_{C^{k+1}}$ is sufficiently small, and $\Phi = \Phi_0 + \Phi_1$ as above with $\|\Phi_1\|_{C^1} \leq 1$, then $v[\phi, \tilde{r}]$ is positive, and even a member of $\mathcal{C}$ with $v[\phi, \tilde{r}](y) = 1$ if $y_2 < \delta$. Thus $T_{\tilde{r}}$ maps $\mathcal{P}_\delta^k$ into itself for $\|\tilde{r}\|_{C^{k+1}}$ small enough. Also, $\Phi_*[0] = 0$, so $v[\phi, 0] \equiv 1$ for any $\Phi \in \mathcal{P}_\delta$. Thus $T_0\Phi = \Phi_0$, and therefore T_0 has a unique fixed point, namely Φ_0. If $\Phi, \Phi' \in \mathcal{P}_\delta^k$

$$
\begin{aligned}
\Phi &= \Phi_0 + \Phi_1 \,, \quad \Phi' = \Phi_0 + \Phi_1' \,, \\
\|\Phi_1\|_{C_1} &\leq 1 \,, \quad \|\Phi_1'\|_{C^1} \leq 1 \,,
\end{aligned}
$$

then it is easy to estimate

$$\|\Phi_*[\tilde{r}] - \Phi_*'[\tilde{r}]\|_{C^k(R)} \leq \|\Phi - \Phi'\|_{C^k} \|\tilde{r}\|_{C^{k+1}}$$

$$\|v[\Phi, \tilde{r}] - v[\Phi', \tilde{r}]\|_{C^k} \leq V \|\tilde{r}\|_{C^{k+1}} \|\Phi_1 - \Phi_1'\|_{C^k}$$

for V depending on δ.

Next we need a few facts about the mapping from a velocity c to its travel-time function $\tau(x, y)$. We will assume that $x_2 = 0$ and $c(y) \equiv 1$ for $y_2 < \delta$. Then for $|x - y| < \delta$, $\tau(x, y) = |x - y|$. For y with $y_2 > \delta$, if $c \in \mathcal{C}$ then $\tau(x, y)$ is a C^1 solution of the eikonal equation

$$c^2 |\nabla \tau|^2 = 1 \ .$$

If moreover

$$c \in \mathcal{C}_M^0 = \left\{ c \in \mathcal{C} : \| \log c \|_{c^0(R)} \leq M \right\}$$

then we obtain the obvious estimate

$$\|\tau\|_{C^1(R)} \leq C_M^0$$

for a suitable constant $C_M^0 > 0$.

We also need bounds on the second derivatives of τ, which can be got as follows. The Hamiltonian system associated to the eikonal equations is

$$\dot{y} = c^2 \eta \quad \dot{\eta} = -c \nabla c |\eta|^2 = -\frac{\nabla c}{c} \ .$$

Denote by $(y(t, y_1), \xi(t, y_1))$ the solution starting (at $t = 0$) at

$$\left(y_1, \delta, \frac{x_1 - y_1}{|x - y|} , \frac{-y_2}{|x - y|} \right) \ .$$

Then

$$\begin{aligned}
\tau(y(t, y_1), x) &= \tau((y_1, \delta), x) + t \\
\nabla \tau(y(t, y_1), x) &= \eta(t, y_1) \ .
\end{aligned}$$

From differentiation of the eikonal equation

$$\nabla \tau \cdot \nabla \frac{\partial \tau}{\partial x_i} = -2 c^{-3} \frac{\partial c}{\partial x_i}$$

we see that the derivative of $\frac{\partial \tau}{\partial x_i}$ *along the rays* is bounded by $\|c\|_1$. On the other hand,

$$\begin{aligned}
\left(\frac{\partial y}{\partial y_1} \right)^{\cdot} (t, y_1) &= \frac{\partial}{\partial y_1} (c^2(y(t, y_1)) \eta(t, y_1)) \\
&= 2 c(y(t, y_1)) \left[\nabla c(y(t, y_1)) \cdot \frac{\partial y}{\partial y_1}(t, y_1) \right] \cdot \eta(t, y_1) \\
&\quad + c^2(y(t, y_1)) \frac{\partial \eta}{\partial y_1}(t, y_1) \\
\left(\frac{\partial \eta}{\partial y_1} \right)^{\cdot} (t, y_1) &= \frac{\partial}{\partial y_1} (\nabla \log c(y(t, y_1))) \\
&= \nabla \nabla \log c(y(t, y_1)) \cdot \frac{\partial y}{\partial y_1}(t, y_1)
\end{aligned}$$

whence a standard O.D.E. estimate gives

$$\left| \frac{\partial y}{\partial y_1}(t, y_1) \right| \leq \text{const.} \, \|\log c\|_{C^2(R)}$$

when $y(t, y_1) \in R$. The assumption that no conjugate points exist implies that $(t, y_1) \mapsto y(t, y_1)$ is a global coordinate system in R, so $\frac{\partial y}{\partial y_1}$ and $\dot{y}$ are linearly independent. Moreover, it is possible to show that the condition number of

$$\begin{pmatrix} \frac{\partial y}{\partial y_1} \\ \dot{y} \end{pmatrix}$$

is

$$\leq C(1 - t\|\nabla \log c\|_{C^1})^{-1}$$

(this is an explicit bound guaranteeing global geodesic coordinates). Since

$$\left(\frac{\partial \eta}{\partial y_1} \right)^{\cdot} = \left(\nabla \nabla \tau \cdot \frac{\partial y}{\partial y_1} \right)^{\cdot} = \nabla \nabla \log c \cdot \frac{\partial y}{\partial y_1}$$

the condition number estimate shows that

$$\|\nabla \nabla \tau\|_{C^0} \leq C \left[(1 - t\|\nabla \log c\|_{C^1})^{-1} \|\nabla \log c\|_{C^1} + 1 \right] .$$

Define

$$\mathcal{C}^2_{M,P} = \left\{ c \in \mathcal{C}^0_M : \|\nabla \log c\|_{C^1} \leq P \right\} .$$

Then if P is small enough, there is a constant $K_2 > 0$ so that

$$c \in \mathcal{C}^2_{M,P} \Rightarrow \|\phi(c; \cdot, \cdot) - \Phi_0\|_{C^2} \leq K_2 .$$

Next, suppose $c_2, c_2 \in \mathcal{C}^2_{M,P}$. Then

$$\nabla \tau_1 \cdot \nabla \frac{\partial \tau_1}{\partial x_i} - \nabla \tau_2 \cdot \nabla \frac{\partial \tau_2}{\partial x_i} = 2 \left(c_2^{-3} \frac{\partial c_2}{\partial x_i} - c_1^{-3} \frac{\partial c_1}{\partial x_i} \right)$$

$$= \nabla \tau_1 \cdot \nabla \left(\frac{\partial \tau_1}{\partial x_i} - \frac{\partial \tau_2}{\partial x_i} \right) + (\nabla \tau_1 - \nabla \tau_2) \cdot \nabla \frac{\partial \tau_2}{\partial x_i} .$$

Integrating along the rays for τ_1 and using $\tau_1 = \tau_2$ at $y_2 = \delta$ we get

$$\|\tau_1 - \tau_2\|_{C^1} \leq \frac{1}{2} L \|c_1 - c_2\|_{C^1}$$

where L is a function of M and P (and in particular of $\|\nabla \log c_i\|_{C^1}$), whence

$$\|\phi(c_1; \cdot, \cdot) - \phi(c_2; \cdot, \cdot)\|_{C^1} \leq L \|c_1 - c_2\|_{C^1} .$$

Now we are ready to string the various facts together. Set

$$\mathcal{B} = \mathcal{P}^2_\delta \cap \{\|\Phi - \Phi_0\|_{C^2} \leq K_2\} .$$

For $\Phi \in \mathcal{B}$,

$$
\begin{aligned}
\|v[\Phi, \tilde{r}] - 1\|_{C^2} &\leq \|\tilde{v}[\Phi, \tilde{r}] - \tilde{v}[\Phi_0, \tilde{r}]\|_{C^2} + \|\tilde{v}[\Phi_0, \tilde{r}] - \tilde{v}[\Phi_0, 0]\|_{C^2} \\
&\leq VK_2\|r\|_{C^3} + \|\Phi_*[\tilde{r}]\|_{C^2} \\
&\leq (VK_2 + W)\|r\|_{C^3}
\end{aligned}
$$

where W is an upper bound for the length of the curves $x_r \mapsto (x_r, \Phi(y, x_r))$ as y ranges over R and Φ over $\mathcal{B}$.

Consequently there exists $Q > 0$ so that if $\|\tilde{r}\|_{C^3} \leq Q$, then $\Phi \in \mathcal{B}$ implies

$$
v[\Phi, \tilde{r}] \in \mathcal{C}^2_{M,P} .
$$

Q depends on V, M, and P (and on K_2 and W, though these depend on M and P.) Consequently, if $\Phi \in \mathcal{B}, \|r\|_3 \leq Q$, then

$$
\|\phi(v[\Phi, \tilde{r}]; \cdot, \cdot) - \Phi_0\|_{C^2} \leq K_2 ,
$$

i.e. $T_{\tilde{r}}$ maps $\mathcal{B}$ into itself. Next note that for $\Phi_1, \Phi_2 \in \mathcal{B}$,

$$
\begin{aligned}
\|T_{\tilde{r}}[\Phi_1] - T_{\tilde{r}}[\Phi_2]\|_{C^1} &\leq L\|v[\Phi_1, \tilde{r}] - v[\Phi_2, \tilde{r}]\|_{C^1} \\
&\leq VL\|\tilde{r}\|_{C^2}\|\Phi_1 - \Phi_2\|_{C^1} .
\end{aligned}
$$

Thus by making Q possibly smaller yet, we can arrange that $T_{\tilde{r}}$ is a contraction on $\mathcal{B}$ — but only in the sense of C^1, not C^2. So we don't get a solution of the fixed point problem in $\mathcal{B}$ — nonetheless, the sequence of iterates $\{T_{\tilde{r}}^n[\Phi_r]\}$ is

(a) bounded in C^2

(b) convergent in C^1 to a C^1 solution of $T_{\tilde{r}}\Phi = \Phi$.

This result is only a glimpse of the actual state of affairs concerning the "multi-dimensional travel time transformation," which the author expects to occupy a central position in the study of wave propagational inverse problems.

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